Sheaves and the Serre-Swan Theorem

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Motivation

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The Grunt Work

In order to start thinking about sheaves in more generality we need to first package our topology so we can use it. Clearly we need to open sets but then whenever we have a smaller set we can restrict our function to that set so we need to remember the containment[1]

Definition (Category of Open Sets)

For a topological space X we define the frame of open sets $\mathrm{Ouv}(X)$ to be the category whos objects are open sets of X and for any two open sets

$$\hom_{\mathrm{Ouv}(X)}(U, V) = \begin{cases} \{U \hookrightarrow V\} & U \subseteq V \\ \varnothing & otherwise \end{cases}$$

If you're not comfortable with categories, don't worry just think of it as a big network where we draw arrows between U, V if $U \subseteq V$.

[1] These are the natural place to do sheaf theory, however it is possible to do it in more generality with Grothendiek Topologies

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In the case of functions on a space we would take this category and for each open set take the set of all functions on that set and then whenever we go to a smaller set we have a restriction map sending functions to their restrictions. Since we have all of this information in the category, we can use this category to define a generalisation of this.

Definition (Presheaf)

A presheaf of sets on X is a functor $\mathcal{F} : \mathrm{Ouv}(X) \to \mathbf{Set}^{op}$. That is for each open set V we associate a set $\mathcal{F}(V)$ and for each $V \subseteq U$ we associate a map $res_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$, analagous to the restriction map. This map also must satisfy that for $V \subseteq U \subseteq W$ we have $res_{U,V} \circ res_{V,W} = res_{U,W}$ (note that we can replace sets in this definition with any category we want ie sheaves of rings or sheaves of groups)

We have a problem here however, these maps need not look anything like restrictions. All we've done thus far is say that restricting twice is the same as restricting once. Functions are nice since they have this gluing property. Ie if two functions, defined on two open sets, agree on the intersection then there is a unique function that agrees with both of them

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Definition (Gluing Condition)

For an open set U and an open cover $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. If we have a sequence of elements $f_\lambda \in U_\lambda$ such that on each $U_\lambda \cap U_\mu, res_{U_\lambda \cap U_\mu, U_\lambda}(f_\lambda) = res_{U_\lambda \cap U_\mu, U_\mu}(f_\mu)$ Then there is some f such that $f_\lambda = res_{U_\lambda, U}(f)$. We can write this categorically by saying that the following diagram is an equaliser

$$\mathcal{F}(U) \to \prod_{\lambda \in \Lambda} \mathcal{F}(U_\lambda) \rightrightarrows \prod_{\lambda, \mu \in \Lambda} \mathcal{F}(U_\lambda \cap U_\mu)$$

Where the two maps are induced by $f_\lambda \mapsto res_{U_\lambda \cap U_\mu, U_\lambda}(f_\lambda)$ and $f_\lambda \mapsto res_{U_\lambda \cap U_\mu, U_\mu}(f_\mu)$

Definition (Sheaves)

A sheaf of sets is a presheaf of sets that satisfies the gluing condition. By replacing sets and set functions to rings and ring homomorphisms we get a sheaf of rings, or with abelian groups and group homomorphisms we get sheaves of abelian group etc

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We have a miriad of examples of different sheaves.

1. For any topological space X and any set S we have the constant sheaf of sets $\mathcal{F}(U) = S$ where $res_{U,V} = id$
2. For any topological spaces X, Y we have the sheaf of continuous functions, $\mathcal{F}(U) = \{f : U \to Y | f \text{ cts}\}$ with $res_{U,V}(f) = f|_U$
3. In fact if Y has an additional ring or group structure we can make this a sheaf of rings or groups. For example the sheaf of rings on a space X given by $\mathcal{F}(U) = \{f : U \to \mathbb{R} | f \text{ cts}\}$ with the natural restrictions
4. Given a manifold M we can construct an analagous sheaf but since manifolds are differentiable structures we can ask that the sheaf sees this too. Ie we can make the sheaf $\mathcal{O}_M(U) = \{f : U \to \mathbb{R} | f \text{ smooth}\}$
5. Given an affine variety V we can encapsulate the algebraic structure by constructing the sheaf $\mathcal{O}_V(U) = \{f | U \subset \operatorname{dom}(f), f \text{ regular}\}$
6. For a pair of spaces X, Y with a projection map $\pi : X \to Y$ we can define the sheaf of sections on Y. That is $\mathcal{O}_{X \to Y}(U) = \{s : U \to X | \pi \circ s = id\}$

Doing Geometry

We now need to introduce what is to me, the main point of this. Sheaves give a natural structure to a topological space. One problem is that sets are somewhat devoid of structure so in order to have structure we want to give a space something stronger than a sheaf of sets. Instead we give it a sheaf of rings

Definition (Ringed Space)

A ringed space $(X, \mathcal{O}_X)$ is a pair of a space X and a sheaf of rings on that space $\mathcal{O}_X$ called the structure sheaf of this space.

Although, it again isn't clear that this summarises the structure of functions on a space to do so we want to add just one more condition

Remembering the case of $\mathcal{O}_M$ one thing we might want to look at is the derivative of these functions. When we do so however we only really care about what the function looks like at a point. Since lots of functions are the same when we look locally we want to consider our functions modulo being the same on some neighborhood of a point. In the case of manifolds we call these "germs"

Definition (Stalk)

For a sheaf $\mathcal{F}$ on some space. We define the stalk at some point p, $\mathcal{F}_p = \{(f, U) | f \in \mathcal{F}(U)\}/\sim$ where $(f, U) \sim (g, V)$ if there is some open $p \in W \subset U \cap V$ so that $res_{W,U}(f) = res_{W,V}(g)$. Written categorically this is just

$$\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U)$$

This describes all possible functions near p.

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It's worth spending some time trying to figure out what the ring of functions near a point look like in the case of our nice ringed spaces. For example if we take the ringed space $(M, \mathcal{O}_M)$ for a smooth manifold M then we can look at the stalk at some point p. This is the ring of all germs of functions defined at p. By some continuity argument we see that any function f that doesn't evaluate to zero at a point will have some neighborhood around it that doesn't evaluate to zero. And so we can define $1/f$ in some neighborhood of p. So a germ is invertable if and only if it evaluates to zero at a point. This means the ring $\mathcal{O}_{M,p}$ has a maximal ideal given by

$$\{f \in \mathcal{O}_{M,p} | f(p) = 0\}$$

So in fact the ringed space $M, \mathcal{O}_M$ will always have local stalks and this will be true for any ringed space that looks reasonably like some set of functions onto a ringed space

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Definition (Locally ringed space)

A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that the stalks at every point are local. Note that this is (locally ring)ed not (locally) ringed

These objects summarise all that we care about when we look at functions on a space, which means they encapsulate geometry and so lie at the intersecton of differential and algebraic geometry

1. For any manifold $(M, \mathcal{O}_M)$ is a locally ringed space. We could even take this as our definition of a manifold, ie a locally ringed space locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}}^n)$
2. For any variety $(V, \mathcal{O}_V)$ is a locally ringed space since $\mathcal{O}_{V,p} = k[V]_{\langle x_i - p_i \rangle}$
3. For any ring A we can take the space $\operatorname{spec} A$ and give it the sheaf

$$\mathcal{O}_{\operatorname{spec} A} = \{s : U \to \bigsqcup_{p \in \operatorname{spec} A} A_p | s(p) \in p \text{ and LAF}\}$$

   Where LAF means that s is locally a fraction, ie on some neigborhood its equal to a/b
4. Given a locally ringed space X we say that X is a scheme if it's locally isomorphic to $(\operatorname{spec} A, \mathcal{O}_{\operatorname{spec} A)}$ for some collection of As

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Serre-Swan

Theorem (Differential Geometry)

For a manifold $(M, \mathcal{O}_M)$ the global sections functor is a categorical equivalence between the category of vector bundles over M and the category of finitely generated projective modules over $C^\infty(M)$

Theorem (Algebraic Geometry)

For an affine variety X there is a categorical equivalence between the category of vector bundles over X and the category of finitely generated projective modules over $\Gamma(X)$

Theorem (Topology)

For a compact, Hausdorff space X letting $C(X)$ be the ring of continuous functions $X \to \mathbb{R}$ or $\mathbb{C}$. There is a categorical equivalence between the category of $\mathbb{R}$or $\mathbb{C}$-vector bundles over X to the category of finitely generated projective modules over $C(X)$

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Theorem (Generality)

[1]For a locally ringed space $(X, \mathcal{O}_X)$. If $\mathcal{O}_X - \mathbf{mod}$ contains an admissible subcategory $\mathcal{C}$, and that every locally free sheaf of finite rank is finitely generated by global sections. Then the global sections functor induces an equivalence between the category of vector bundles over X and the category of projective modules over $\mathcal{O}_X(X)$

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Thanks for coming! If you have any questions you can email me at Kyle.Thompson@warwick.ac.uk

References I

[1] A. S. Morye, "Note on the Serre-Swan theorem," *arXiv (Cornell University)*, Jan. 2009. DOI: 10.48550/arxiv.0905.0319. [Online]. Available: https://arxiv.org/abs/0905.0319.